PMM U.S.S.R.,Vol.44,pp.375-378 0021-8928/81/3 0375 $7.50/0

UDC 539.383

THE CONTACT PROBLEM FOR A HEAVY HALF-PLANE*

V. M. ALEKSANDROV and L. M. FILIPPOVA

The problem of a smooth rigid stamp penentration into an elastic half-plane is considered with allowance for initial stresses produced by the half-plane own weight. An integral equation is derived for the contact pressure, whose solution is obtained by the asymptotic method. It is shown that in this formulation the stamp displacement is uniquely determined, unlike in the classical problem of half-plane contact, where it is not determined.

1. Let us consider a half-plane of incompressible isotropic elastic material subjected to forces of its own weight. These forces produce in the half-plane a hydrostatic state of stress

$$\sigma_{11}^\circ = \sigma_{22}^\circ = \sigma_{33}^\circ = -\gamma_0 x_2 \tag{1.1}$$

where γ_0 is the specific weight and x_2 a coordinate taken from the half-plane boundary. By virtue of the material incompressibility it is undeformed in this state of stress.

A small plane deformation is produced in this initial state by the action of a smooth rigid stamp at the half-plane boundary. The general equations defining small deformations of a prestressed incompressible body are of the form /1/

$$\frac{\partial \theta_{sk}}{\partial x_s} = 0, \quad \theta_{sk} = \sigma_{sk}^* - \sigma_{ms}^\circ \frac{\partial u_m}{\partial x_k}, \quad \frac{\partial u_m}{\partial x_m} = 0 \tag{1.2}$$

where σ_{ms}° are initial stresses, x_k are Cartesian coordinates of the body which in the initial state of stress is assumed undeformed, u_s are components of the additional displacement vector, and σ_{sk}^* are small stress increments due to these. In the case considered here of plane deformation all subscripts in (1.2) assume the values 1 and 2, and recurrent subscripts imply summation. The last of Eqs. (1.2) represents the condition of incompressibility.

For calculating the quantities σ_{mk}^* we use the equation of state of an isotropic incompressible material under finite deformations /1/

$$\sigma_{mn} = \alpha_1 M_{mn} - \alpha_2 N_{mn} + \sigma\delta_{mn}, \quad M_{sk} = \frac{\partial X_s}{\partial x_t}\frac{\partial X_k}{\partial x_t}, \quad N_{sk} M_{kt} = \delta_{st} \tag{1.3}$$

where X_s are Cartesian coordinates of the /body in the/ deformed state, M_{sk} are components of the Finger deformation measure, N_{mn} are components of the tensor inverse of the Finger measure, δ_{st} is the Kronecker delta, σ is the pressure, and α_1 and α_2 are some functions of deformation invariants. Setting

$$X_s = x_s + u_s, \quad \sigma_{mn} = \sigma_{mn}^\circ + \sigma_{mn}^*, \quad \sigma = \sigma^\circ + \sigma^*, \quad \alpha_1 = \alpha_1^\circ + \alpha_1^*, \quad \alpha_2 = \alpha_2^\circ + \alpha_2^*$$

and linearizing (1.3) in the neighborhood of the initial state $X_s^\circ = x_s$ we obtain

$$\sigma_{mn}^* = \mu\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right) + q^*\delta_{mn}, \quad \mu = \alpha_1^\circ + \alpha_2^\circ, \quad q^* = \alpha_1^* - \alpha_2^* + \sigma^* \tag{1.4}$$

For the Mooney material /1/ we have $\alpha_1 = 2C_1 = \text{const}$ and $\alpha_2 = 2C_2 = \text{const}$ from which $\mu = 2(C_1 + C_2)$.

Using the notation $x_1 = x;\ x_2 = y;\ u_1 = u,\ u_2 = v,\ \gamma_0 = \mu\gamma$, and $q^* = \mu q$, from (1.1), (1.2), and (1.4) we obtain the following system of equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}(\gamma v + q) = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial y}(\gamma v + q) = 0 \tag{1.5}$$

Let us consider the problem of action of a single normal concentrated force on the boundary of a heavy half-plane. The boundary conditions for this problem are ($\delta(x)$ is the delta function)

$$y = 0, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} + \frac{q}{2} = -\frac{1}{2\mu}\delta(x) \tag{1.6}$$

*Prikl.Matem.Mekhan.,44,No.3,535-539,1980

We introduce the new unknown function $p = \gamma v + q$ and, applying to (1.5) along the x-coordinate the Fourier transform, we obtain

$$\bar{u}'' - \alpha^2\bar{u} - i\alpha p = 0,\ \bar{v}'' - \alpha^2\bar{v} + p = 0,\ \bar{v}' - i\alpha\bar{u} = 0, \quad \bar{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx \tag{1.7}$$

where the prime denotes the derivative with respect to the y-coordinate. The transformed boundary conditions assume the form

$$\bar{u}' - i\alpha\bar{v} = 0, \quad \bar{v}' - \frac{1}{2}\gamma\bar{v} + \frac{1}{2}\bar{p} = -\frac{1}{2\mu\sqrt{2\pi}} \tag{1.8}$$

The solution of system (1.7) that satisfies conditions (1.8) at $y = 0$ and the conditions of damping as $y \to \infty$, is of the form

$$\bar{u} = \frac{i\alpha y e^{-|\alpha|y}}{\sqrt{2\pi}\,\mu(2|\alpha| + \gamma)}, \quad \bar{v} = \frac{(1 + |\alpha|y)e^{-|\alpha|y}}{\sqrt{2\pi}\,\mu(2|\alpha| + \gamma)}, \quad p = -\frac{2|\alpha|^{-|\alpha|y}}{\sqrt{2\pi}\,\mu(2|\alpha| + \gamma)} \tag{1.9}$$

Using the Fourier transform we obtain the expression for the function $v(x, 0)$ which represents the kernel of the integral equation of the contact problem. From the tables /2/ we obtain

$$v(x, 0) = -\frac{1}{2\pi\mu}\left[\operatorname{si}\left(\frac{1}{2}\gamma|x|\right)\sin\left(\frac{1}{2}\gamma|x|\right) + \operatorname{ci}\left(\frac{1}{2}\gamma|x|\right)\cos\left(\frac{1}{2}\gamma|x|\right)\right] \tag{1.10}$$

$$\operatorname{si}(z) = -\int_z^{\infty}\frac{\sin t}{t}dt, \quad \operatorname{ci}(z) = -\int_z^{\infty}\frac{\cos t}{t}dt$$

In a similar problem, without allowance for initial stresses produced by the weight (the Flamant problem), we have /1/

$$v(x, 0) = \frac{1}{2\pi\mu}\ln\frac{1}{|x|} + D \tag{1.11}$$

where D is an undefined constant.

Using the asymptotic representation of functions $\operatorname{si}(z)$ and $\operatorname{ci}(z)$ as $z \to \infty$ /3/, from (1.10) we obtain

$$v(x, 0) \approx \frac{2}{\pi\mu}\frac{1}{(\gamma x)^2}, \quad |x| \to \infty$$

Thus, when the weight of the half-plane is taken into account, the displacements produced in it by the concentrated force are, unlike in (1.11), uniquely determined and decrease at infinity.

For small γ from (1.10) we obtain /3/

$$v(x, 0) = -\frac{1}{2\pi\mu}C - \frac{1}{2\pi\mu}\ln\left(\frac{1}{2}\gamma\right) + \frac{1}{2\pi\mu}\ln\frac{1}{|x|} + O(\gamma)$$

where C is Euler's constant. This shows that as $\gamma \to 0$ the solution becomes the classical one(1.11) and $D = \infty$.

The boundary conditions at $y = 0$ of the problem of rigid smooth stamp penetration into the half-plane, considered below, are of the form

$$v = f_1(x), \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \quad |x| \leqslant a, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \quad \frac{\partial u}{\partial y} + \frac{q}{2} = 0, \quad |x| \geqslant a \tag{1.12}$$

where $2a$ is the stamp width and $f_1(x)$ is a function that defines the stamp shape.

2. Formulas (1.10) and (1.12) yield for the contact pressure $2\mu\varphi(x)$ the following integral equation:

$$\int_{-1}^{1} K(t - \tau)\varphi(t)\,dt = \pi f(\tau), \quad |\tau| \leqslant 1 \tag{2.1}$$

$$K(t - \tau) = -\operatorname{si}\left(\frac{|t-\tau|}{\lambda}\right)\sin\left(\frac{|t-\tau|}{\lambda}\right) - \operatorname{ci}\left(\frac{|t-\tau|}{\lambda}\right)\cos\left(\frac{|t-\tau|}{\lambda}\right) = -\ln\frac{|t-\tau|}{\lambda} + \frac{\pi}{2}\frac{|t-\tau|}{\lambda} - C + F\left(\frac{t-\tau}{\lambda}\right)$$

$$F(z) = \ln|z|F_1(z) + |z|F_2(z) + F_3(z), \quad \lambda = \frac{2}{a\gamma}, \quad \tau = \frac{x}{a}, \quad f(\tau) = \frac{1}{a}f_1(a\tau)$$

As $z \to 0$ function $F_i(z)$ becomes of order z^2. Equation (2.1) must be supplemented by the condition of balancing the stamp by the external force $Q = 2\mu aP$

$$P = \int_{-1}^{1}\varphi(t)\,dt \tag{2.2}$$

The method of solving equations of the (2.1) type has been worked out in /4/, where it is shown that for large values of the parameter λ the solution can be represented in the form

$$\omega(\tau) = \omega_0(\tau) + \lambda^{-1}\omega_1(\tau) + o(\lambda^{-2-\varepsilon}), \quad \omega(\tau) = \varphi(\tau)\sqrt{1-\tau^2} \tag{2.3}$$

where $\varepsilon \neq 0$ is an arbitrarily small positive number.

Note that in real cases the parameter λ^{-1} is very small. For instance, when $\gamma_0 = 10^2$ dyn/cm^2, $2\mu = 10^{11}$ dny/cm^2, and $a = 10^3$cm, we obtain $\lambda^{-1} = 10^{-6}$. Not too small λ^{-1} are theoretically possible in the case of superheavy material with very low shearing modulus and for very wide stamps. Owing to the smallness of parameter λ^{-1}, we restrict in subsequent calculations the solution of Eq. (2.1) to terms of order λ^{-1}. As shown in /4/, the function $F(z)$ in the expression of the kernel of (2.1) does not affect functions $\omega_0(\tau)$ and $\omega_1(\tau)$, it is, therefore, possible to set $F = 0$ without affecting the above accuracy. We would, however, point out that the calculation of several subsequent terms of expansion using the method of /4/ does not present any fundamental difficulties.

For an inclined stamp with a plane face we have

$$f_1(x) = \delta + kx, \quad f(\tau) = a^{-1}b + k\tau$$

where δ is the stamp displacement. Applying the method of /4/ we obtain

$$\omega(\tau) = \frac{1}{\pi}P + k\tau + \lambda^{-1}\left[\frac{2}{\pi^2}PS_1(\tau) + \frac{2k}{\pi}\tau - \frac{k}{\pi}(1-\tau^2)\ln\frac{1-\tau}{1+\tau}\right] + o(\lambda^{-2-\varepsilon}) \tag{2.4}$$

$$S_1(\tau) = (1-2\tau^2) + 2\sqrt{1-\tau^2}\sum_{k=1}^{\infty}\frac{\sin[(2k+1)\arccos\tau]}{(2k+1)^2}$$

The stamp displacement determined by formula

$$\int_{-1}^{1} K(t)\varphi(t)\,dt = \pi f(0) \tag{2.5}$$

is of the form

$$\delta = \frac{Q}{2\pi\mu}(\ln 2\lambda - \mathrm{C} + 0.4053\pi\lambda^{-1}) + o(\lambda^{-2-\varepsilon}) \tag{2.6}$$

Formula (2.6) shows that the allowance for stresses produced by the /material/ weight uniquely determines the stamp displacement, unlike in the classical contact problem for the half-plane, where, as shown in /1,5/, it remains undefined. As $\lambda \to \infty$ the stamp pressure distribution approaches in conformity with (2.4) the classical solution /5/, and $\delta \to \infty$. Let us now consider the parabolic stamp

$$f_1(x) = \delta - Ax^2, \quad f(\tau) = a^{-1}(\delta - Aa^2\tau^2)$$

The solution of Eq. (2.1) is

$$\omega(\tau) = \frac{P}{\pi} + Aa(1-2\tau^2) + \lambda^{-1}\left[\frac{2P}{\pi^2}S_1(\tau) + \frac{4a}{\pi}S_4(\tau)\right] + o(\lambda^{-2-\varepsilon}) \tag{2.7}$$

$$S_4(\tau) = \frac{1}{3} + (1-2\tau^2) + \tau(1-\tau^2)\ln\frac{1-\tau}{1+\tau}$$

Formula (2.5) yields

$$\pi a^{-1}\delta = P(\ln 2\lambda - \mathrm{C}) + 1.571Aa + \lambda^{-1}\pi(0.4053P - 0.1415Aa) + o(\lambda^{-2-\varepsilon}) \tag{2.8}$$

In the case of a parabolic stamp the contact zero width $2a$ is not a priori known, and is to be determined by the condition of boundedness of contact pressure at the contact zone boundary $\omega(\pm 1) = 0$. In dimensional quantities this condition, accurate to terms of order λ^{-1} is of the form

$$\frac{Q}{2\mu} - \pi Aa^2 = \gamma\left(\frac{Q}{2\pi\mu}a + 0.3334Aa^3\right) \tag{2.9}$$

whose solution is

$$a = a_0(1 - 1.3334\pi^{-1}\lambda_0^{-1}) + o(\lambda_0^{-2-\varepsilon}), \quad a_0 = \sqrt{\frac{Q}{2\pi\mu A}}, \quad \lambda_0 = \frac{2}{a_0\gamma} \tag{2.10}$$

Taking into account (2.10), instead of (2.7) we obtain

$$2\mu\omega(\tau) = \frac{2Q}{\pi a_0}\left\{(1-\tau^2) + \lambda_0^{-1}\left[1.3334\tau^2 + S_1(\tau) + \frac{1}{2}S_4(\tau)\right]\right\} + o(\lambda_0^{-2-\varepsilon}) \tag{2.11}$$

The substitution of (2.10) into (2.8) yields

$$\delta = \frac{Q}{2\pi\mu}(\ln 2\lambda_0 - 0.0742 + 1.1319\lambda_0^{-1}) + o(\lambda_0^{-2-\varepsilon}) \tag{2.12}$$

Formulas (2.10) — (2.12) show that as $\gamma \rightarrow 0$, the constact pressure distribution approaches that given in /1,5/, depending on the contact zone dimension and on the force, while the stamp penetration increases indefinitely.

The authors thank L. M. Zubov for valuable advice.

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Translated by J.J.D.